Small Wheel Bicycle Model Derivation and Balancing Controller Design

Zhidi Yang

July 2018

Contents

1 Abstract

This article develop an electronically controlled bicycle in simulation. It derives a simplified bicycle model, which can be used to design a linear or non-linear controller for autonomous bicycle. The accuracy of this model is acceptable when the bicycle is moving in relative low speed and the handlebar is not steering very fast. Two controllers are implemented in simulations and the sliding mode control controller is superior than LQR on performance and robustness.

2 Introduction

We would like to build an electronically controlled bicycle. This could be a robotic bicycle, or a 'fly-by-wire' bicycle in which the balance is managed electrically, with no direct mechanical connection between the rider and the steering. We would like to develop a controller for such a bicycle in simulation. Because optimizing a controller might take thousands or millions of simulations, and because we might want to employ theoretical control theory, it is useful to have moderately simple equations of motion. Our choices from the literature are reviewed in Meijaard et al.(2007). In short, interpretable equations are generally only linear, and we want a non-linear (large-amplitude) control. And non-linear equations are complicated and not even expressible in closed form. Also, the lack of a pure analytic form of the governing equations in the benchmark can make controller design more difficult. This paper attempts to derive a pure analytical symbolic expression of a simplified model that can be used to design linear or nonlinear controllers for electronically controlled bicycle.

But the robotic bicycle has advantages, as far as governing equations go. We can assume that the steering is controlled separately in an inner, and faster, control loop. Thus, we don't need the steer dynamics equations. And the lean equations can, we think, be well approximated by a simpler model. The goal of this paper is to describe such a simpler model and possible controls using this model. We compared the accuracy of the model to the benchmark of Meijaard et al.(2007). We implemented the controller in the simulation and tested its performance and robustness. In this article, we did not consider the self-stability of bicycles, since we assume that the bicycle is balanced by the steering control of the handlebars. The speed of the bike is considered a key parameter and not set to a constant.

In the end, the robustness of the controller can be partially tested by using it in a more complex, and perhaps more realistic, simulation.

3 Related work

Safety bikes are the direct ancestors of most modern bicycles and they have two wheels of the same size[2]. Research on the dynamics of bicycles began in the 19th century and they were surprisingly complicated. Many respected scientists and researchers have studied this, but due to its complexity, different models, different parameters and different variables, only a few of whom have cross-checked[1]. Meijaard et al.(2007) reviewed all of these studies and established benchmarks for bicycle dynamics. They assume an ideal knife-edge rolling point-contact with horizontal ground. Their solution can be used to inspect other models. However, when considering the large lean angle and steering angle, the pitch of the bicycle is the solution of a fourth-order polynomial equation. There is no pure analytical expression to express the full dynamics of bicycle[1]. They also derived a well-defined equation for a safe bicycle model and reveals the stability of a controlled or uncontrolled bicycle. The safety bike has a tilting steering axis and fork offset, which is very important for balance and control[1].

To balance a bicycle, one can balance a leaned forward-moving bicycle by steer to the direction of lean. It can move the ground contact point under the cyclist's position[1]. Due to the various geometry, inertia and gyroscope characteristics of bicycles, some uncontrolled bicycles can also balance themselves at some speed. Since many different methods, different coordinates and different languages can be used to derive the governing equations for bicycles, it is difficult to match all the initial conditions of these different methods for cross-checking. Meijaard et al.(2007) used the Whipple bicycle model, which takes into account all rigid body effects and omits some subtle effects of the wheel. The two wheels in the model are different.

4 Small wheel bicycle dynamic model derivation

The goal of this section is to derive a simplified dynamic model that can be used to design a steering controller. In order to obtain a purely analytical symbolic expression governing equation, the slight pitch of the bicycle can be ignored. We assume that the bike has a very small wheel and the bike does not pitch at all. In addition, we can also assume that the tilt angle of steer axis ζ and fork offset is zero. The front wheel and handlebar will be turned directly by the controller. Therefore, in this simplified bicycle dynamics, the moment of inertia of the front wheels and other geometries, inertia and gyroscopic forces are negligible.

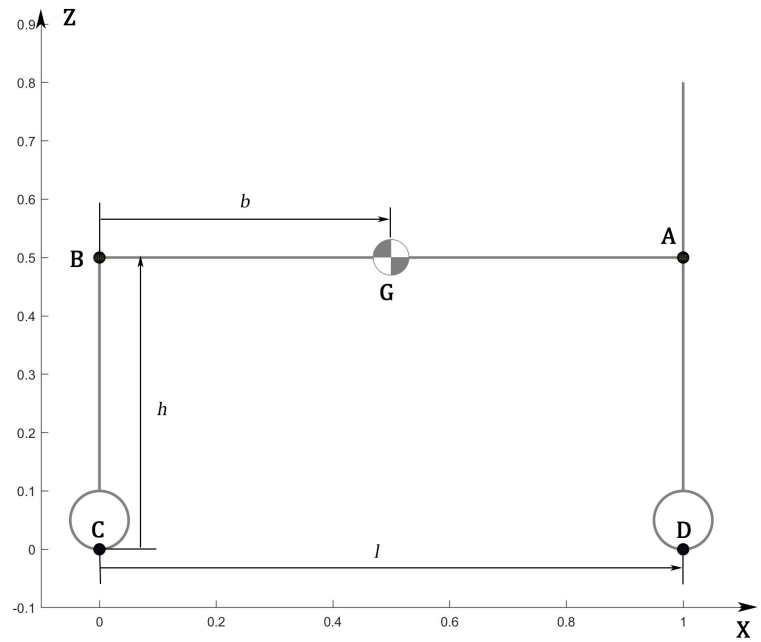

Figure 1: Bicycle model parameters

4.1 Parameters of a bicycle

In the model, the bicycle is divided into two parts. The first part is the rear wheel and the frame. The second part is the front wheel and the front handlebar. The handlebar is rotatable about the front fork steering axis. Two wheels are at xz Plane. Wang and Ruina(2014) defined most of these parameters.

See Figure 1: Point G is the Center of Mass. Point A is the intersection of the frame and the front handlebar. Point B is the point above the rear contact point of the rear frame and it is at the same height of point G. Point C is the contact point of the rear wheel and the ground. Point D is the contact point of the front wheel and the ground. $BG = b$ $BC = h$ $CD = l$

4.1.1 Direction unit vector

$\hat{i}$ is the unit vector in the direction of positive x axis. $\hat{j}$ is the unit vector in the direction of positive y axis. $\hat{k}$ is the unit vector in the direction of positive z axis.

$\hat{\lambda}$ is the unit vector in the direction of the heading of rear wheel. It is on the xy plane. $\hat{n}$ is the unit vector of the normal direction of the heading of rear wheel $\hat{\lambda}$, pointing to the left side of the bicycle. It is also on the xy plane. Both

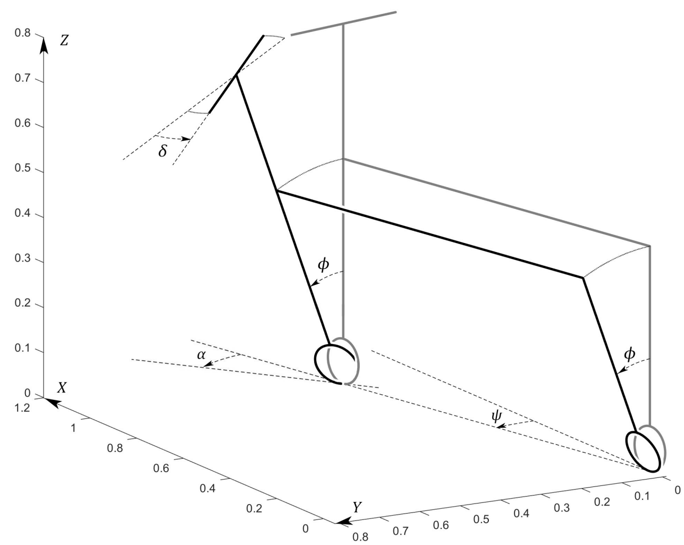

Figure 2: Bicycle dynamic variables

$\hat{\lambda}$ and $\hat{n}$ are perpendicular to $\hat{k}$.

4.1.2 Steering δ

δ is the steering angle (Figure 2). It is the angle of the front handlebar set rotates about the front fork axis. In Figure 2, $\delta = \pi/8$.

4.1.3 Lean ϕ

ϕ is the lean angle (Figure 2). The whole bicycle leans about the connection line between the contact points of front wheel D and rear wheel C. In Figure 2, $\phi = \pi/8$.

4.1.4 Yaw ψ

ψ is the yaw angle (Figure 2). It is the angle between x axis and the heading of rear wheel. The whole bicycle yaws about z axis. The heading of the bicycle $\hat{\imath}$ became $\hat{\lambda}$ and the normal direction of $\hat{\imath}$, which is $\hat{\jmath}$, became $\hat{n}$. Also, the whole bicycle will always lean about the new heading, which is $\hat{\lambda}$. In Figure 2, $\psi = \pi/16$.

4.1.5 Tilt angle of the steer axis ζ

ζ is the tilt angle of steer axis (Figure 3). The tilt angle of steer axis ζ and fork offset contributes to the trail and influence the dynamics of a bicycle. In

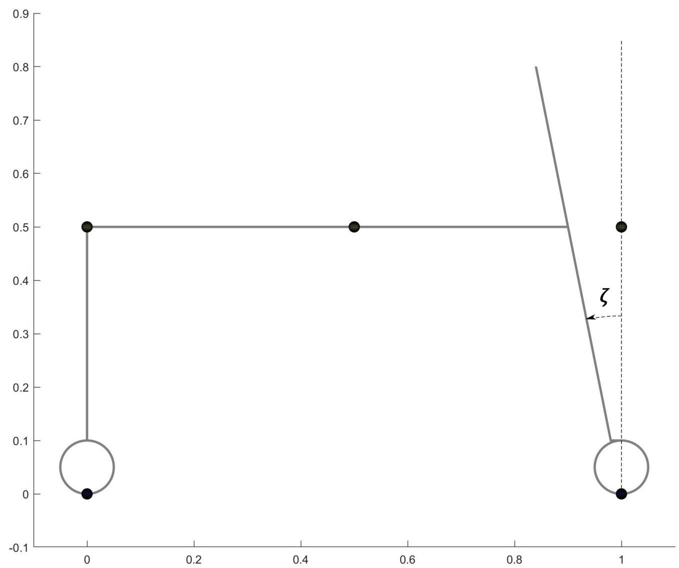

Figure 3: Tilt angle of the steer axis

the governing equation, considering the tilt angle of the steering axis ζ does not increase the accuracy of the model. It will add a constant $\cos\zeta$ term in the part of the equation, which will ruin the simplicity of this model.

4.1.6 Other variables

If the tilt angle of steer axis ζ is not equal to zero, and the steering angle is not equal to zero, the contact point between front wheel and ground will shift on xy plane. Since the distance between front wheel and rear wheel l is much larger than the shift distance. Because steering angle δ and lean angle ϕ are small, we assume the contact point between front wheel and ground is a fix point. It is the contact point of a non-steering upright bicycle. So that the whole bicycle will always leans about $\hat{\lambda}$ axis. Also, since the contact point between front wheel and ground, which is the lowest point on the front wheel, will change due to steering and lean, the whole bicycle will pitch a little bit and the center of mass will be lower than the center of mass a upright bicycle. We assume that the change of height of the center of mass is negligible and the center of mass is always at the position of the center of mass of a non-steering upright bicycle.

4.2 Front wheel direction

α is the angle between the heading of front wheel and the heading of rear wheel $\hat{\lambda}$ (Figure 2). In order to derive the moving direction of front wheel, we need to know the kinematic relationship between α, ϕ and δ. If not considering the tilt

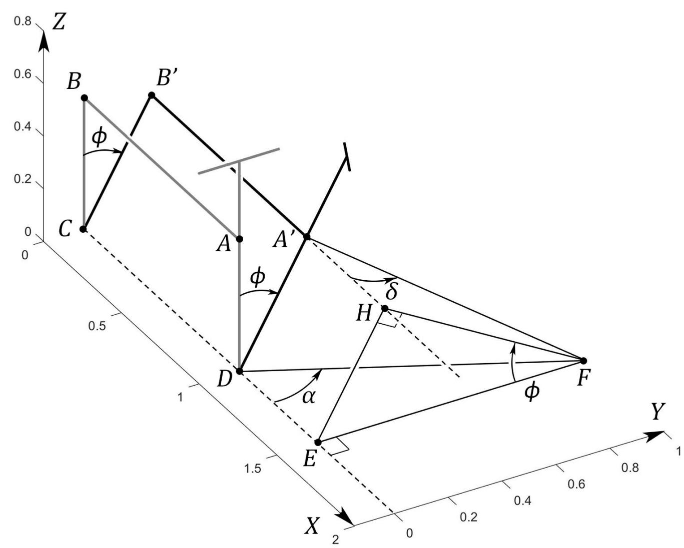

Figure 4: Heading of front wheel

angle of the steer axis, the relationship between α, ϕ and δ can be derived by 3D drawings. Wang and Ruina(2014) derived the following method.

See Figure 4: Assume $ABCD$ is the upright bicycle and $A'B'CD$ is the leaned bicycle. $A'H$ is the direction of the front handlebar in a leaned but not steering bicycle. $A'F$ is the direction of the front handlebar in a leaned and steered bicycle. Point F is on the xy plane and it is on the ground. Point E is also on the xy plane and it is on the x axis. Point E has the same y coordinate of Point F. Line EF is on the xy plane too. Point H is on the extension line of $B'A'$ and plane HEF is perpendicular to $B'A'$. Because the front fork is always perpendicular to the plane of handlebar, $EH \perp A'HF$. Because the front fork is also always perpendicular to the heading of rear wheel, $DE \perp EF$. α is the angle between the heading of the front wheel and the heading of the rear wheel. Based on the spatial relationship, we can get:

$$\frac{FH}{EF} = \cos\phi$$

$$\frac{FH}{A'H} = \tan\delta$$

$$\frac{EF}{DE} = \tan\alpha$$

Since $A'H = DE$

$$\tan\alpha = \frac{\tan\delta}{\cos\phi} \tag{1}$$

If we consider the tilt angle of the steer axis ζ, the relationship between α, ϕ and δ can be derived by 3D rotation. See Appendix A for more details of derivation. The result of relationship between α, ϕ and δ is:

$$\tan\alpha = \frac{\sin\delta\,\cos\zeta}{\cos\delta\,\cos\phi - \sin\delta\,\sin\phi\,\sin\zeta} \tag{2}$$

4.3 Change rate of rear wheel heading yaw angle $\dot{\psi}$ and $\ddot{\psi}$

See Figure 2, the change rate of yaw angle $\dot{\psi}$ is relate to the angle α between the heading of front wheel and the heading of rear wheel. The result accord with the result in Wang and Ruina(2014)'s report.

$$v^{rear} = v_{\hat{\lambda}}^{rear} = v_{\hat{\lambda}}^{front} = v$$

$$v^{front} = \frac{v}{\cos\alpha}$$

$$v_{\hat{n}}^{front} = v^{front} \cdot \sin\alpha = v \cdot \tan\alpha$$

Since

$$v_{\hat{n}}^{front} = \dot{\psi} \cdot l$$

$$\dot{\psi} = \frac{v \cdot \tan\alpha}{l}$$

So that

$$\dot{\psi} = \frac{v\,\tan\delta}{l\,\cos\phi} \tag{3}$$

If considering the tilt angle of the steer axis ζ,

$$\dot{\psi} = \frac{v\,\sin\delta\,\cos\zeta}{l\,(\cos\delta\,\cos\phi - \sin\delta\,\sin\phi\,\sin\zeta)}$$

For $\ddot{\psi}$,

$$\ddot{\psi} = \frac{\partial\dot{\psi}}{\partial v} \cdot \dot{v} + \frac{\partial\dot{\psi}}{\partial\delta} \cdot \dot{\delta} + \frac{\partial\dot{\psi}}{\partial\phi} \cdot \dot{\phi}$$

So that

$$\ddot{\psi} = \frac{\dot{\delta}\,v\,\cos\phi + \dot{v}\,\cos\delta\,\cos\phi\,\sin\delta + \dot{\phi}\,v\,\cos\delta\,\sin\delta\,\sin\phi}{l\cos\delta^2\cos\phi^2} \tag{4}$$

If considering the tilt angle of the steer axis ζ,

$$\ddot{\psi} = \dot{\delta}\left(\frac{v\,\cos\delta\,\cos\zeta}{l\,(\cos\delta\,\cos\phi - \sin\delta\,\sin\phi\,\sin\zeta)} + \frac{v\,\sin\delta\,\cos\zeta\,(\cos\phi\,\sin\delta + \cos\delta\,\sin\phi\,\sin\zeta)}{l\,(\cos\delta\,\cos\phi - \sin\delta\,\sin\phi\,\sin\zeta)^2}\right)$$

$$+\frac{\dot{v}\,\sin\delta\,\cos\zeta}{l\,(\cos\delta\,\cos\phi - \sin\delta\,\sin\phi\,\sin\zeta)} + \frac{\dot{\phi}\,v\,\sin\delta\,\cos\zeta\,(\cos\delta\,\sin\phi + \cos\phi\,\sin\delta\,\sin\zeta)}{l\,(\cos\delta\,\cos\phi - \sin\delta\,\sin\phi\,\sin\zeta)^2}$$

4.4 Acceleration of center of mass

Acceleration of center of mass is derived in Wang and Ruina(2014)'s report. We use the five-terms acceleration equation to rigorously verify it. For the five-term acceleration equation,

$$\vec{a}_p = \vec{a}_{o'/o} + \vec{a}_{p/\beta} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{p/o'}\right) + \dot{\vec{\omega}} \times \vec{r}_{p/o'} + 2\,\vec{\omega} \times \vec{v}_{p/\beta} \tag{5}$$

p is the point we are interested in. o is the origin. $\vec{\omega}$ is the angular velocity vector. For each term in the equation,

$$\vec{v_{o'/o}} = v \cdot \hat{\lambda}$$

$$\vec{a}_{o'/o} = \frac{d}{dt}\vec{v}_{o'/o} = \frac{d}{dt}(v\,\hat{\lambda}) = \dot{v}\,\hat{\lambda} + v\,\dot{\hat{\lambda}} = \dot{v}\,\hat{\lambda} + v\,\vec{\omega} \times \hat{\lambda} = \dot{v}\,\hat{\lambda} + v\,\dot{\psi}\,\hat{k} \times \hat{\lambda} = \dot{v}\,\hat{\lambda} + v\,\dot{\psi}\,\hat{n}$$

$$\hat{e}_r = \cos\phi\,\hat{k} + \sin\phi\,\hat{n}$$

$$\hat{e}_\phi = -\sin\phi\,\hat{k} + \cos\phi\,\hat{n}$$

$$\dot{\hat{e}}_\phi = -\dot{\phi}\,\hat{e}_r$$

$$\vec{v}_{p/\beta} = \dot{\phi}\,r\,\hat{e}_\phi$$

$$\vec{a}_{p/\beta} = \frac{d}{dt}\vec{v}_{p/\beta} = \frac{d}{dt}\left(\dot{\phi}\,r\,\hat{e}_\phi\right) = \ddot{\phi}\,r\,\hat{e}_\phi + \dot{\phi}\,r\,\dot{\hat{e}}_\phi = \ddot{\phi}\,r\,\hat{e}_\phi + \dot{\phi}\,r\,\dot{\phi}\,(-\hat{e}_r) = r\,\ddot{\phi}\,\hat{e}_\phi - \dot{\phi}^2 r\hat{e}_r$$

$$\vec{r}_{p/o'} = b\,\hat{\lambda} + h\,\hat{e}_r$$

$$\vec{\omega} = \dot{\psi}\,\hat{k}$$

$$\dot{\vec{\omega}} = \frac{d}{dt}\left(\dot{\psi}\,\hat{k}\right) = \ddot{\psi}\,\hat{k} + \dot{\psi}\,\dot{\hat{k}} = \ddot{\psi}\,\hat{k}$$

Substitute these terms into five terms acceleration equation (Equation (5)), we can get

$$\begin{aligned} \vec{a}_G = \vec{a}_p = &\left(-b\,\dot{\psi}^2 - 2\,h\,\dot{\phi}\cos\phi\,\dot{\psi} + \dot{v} - h\,\ddot{\psi}\sin\phi\right)\hat{\lambda} + \left(-h\,\sin\phi\,\dot{\phi}^2\right. \\ &\left.-h\,\sin\phi\,\dot{\psi}^2 + v\,\dot{\psi} + b\,\ddot{\psi} + h\,\ddot{\phi}\cos\phi\right)\hat{n} + \left(-h\,\left(\cos\phi\,\dot{\phi}^2 + \ddot{\phi}\sin\phi\right)\right)\hat{k} \end{aligned} \tag{6}$$

The result accord with the result in Wang and Ruina(2014)'s report.

4.5 Moment of inertia of the bicycle

The moment of inertia of the handlebar is relative small compare to rear body and frame assembly of bicycle and we assume that the steering is controlled separately. Thus, we don't need the steer dynamics equations. We assume the moment of inertia of the bicycle is not changing relative to body frame. Most parts of a bicycle are located on the xz plane, so we define the initial moment of inertia of the bicycle $\mathbf{I_0}$ as

$$\mathbf{I_0} = \begin{bmatrix} I_1 & 0 & I_{13} \\ 0 & I_2 & 0 \\ I_{13} & 0 & I_3 \end{bmatrix}$$

For rotation matrix of leaning,

$$\underline{\underline{\mathbf{R}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

The moment of inertia of the bicycle $\mathbf{I}$ is

$$\mathbf{I} = \underline{\underline{\mathbf{R}}} \cdot \mathbf{I_0} \cdot \underline{\underline{\mathbf{R}}}^T$$

$$= \begin{bmatrix} I_1 & I_{13}\,\sin\phi & I_{13}\,\cos\phi \\ I_{13}\,\sin\phi & I_2\cos\phi^2 + I_3\sin\phi^2 & I_3\,\cos\phi\,\sin\phi - I_2\,\cos\phi\,\sin\phi \\ I_{13}\,\cos\phi & I_3\,\cos\phi\,\sin\phi - I_2\,\cos\phi\,\sin\phi & I_3\cos\phi^2 + I_2\sin\phi^2 \end{bmatrix}$$

4.6 Angular momentum balance

For angular momentum balance of an object in 3D,

$$\vec{M}_{/C} = \dot{\vec{H}}_{/C'} \tag{7}$$

c' is a good point. It is a fix point instantaneous at the same position of c.

$$\vec{M}_{/C} = \vec{r}_{G/C} \times m_{tot}\, g\left(-\hat{k}\right) + \vec{r}_{D/C} \times \vec{F}_D \tag{8}$$

$$\vec{H}_{/C'} = \vec{r}_{G/C'} \times m_{tot}\, \vec{v}_G + \vec{H}_{/G}$$

$$\vec{H}_{/G} = \vec{\vec{I}} \cdot \vec{\omega}$$

$$\dot{\vec{H}}_{/C'} = \vec{v}_{G/C'} \times m_{tot}\, \vec{v}_G + \vec{r}_{G/C'} \times m_{tot}\, \vec{a}_G + \dot{\vec{H}}_{/G}$$

Since

$$\dot{\vec{Q}}^{f} = \dot{\vec{Q}}^{\beta} + \vec{\omega} \times \vec{Q}$$

$$\vec{v}_{G/C'} \times m_{tot}\, \vec{v}_G = 0$$

Then

$$\dot{\vec{H}}_{/C'} = \vec{r}_{G/C'} \times m_{tot}\, \vec{a}_G + \vec{\vec{I}} \cdot \dot{\vec{\omega}} + \vec{\omega} \times \left(\vec{\vec{I}} \cdot \vec{\omega}\right) \tag{9}$$

Since

$$\vec{r}_{D/C} = l\,\hat{\lambda}$$

$$\vec{r}_{D/C} \parallel \hat{\lambda}$$

$$\vec{r}_{D/C} \times \vec{F}_D \perp \vec{r}_{D/C}$$

$$\left(\vec{r}_{D/C} \times \vec{F}_D\right) \cdot \hat{\lambda} = 0$$

Computer dot product of $\hat{\lambda}$ and both sides of Equation (7). Substitute Equation (9) and Equation (8) into Equation (7),

$$\vec{M}_{/C} \cdot \hat{\lambda} = \dot{\vec{H}}_{/C'} \cdot \hat{\lambda}$$

$$\vec{r}_{G/C} \times m_{tot}\, g\left(-\hat{k}\right) \cdot \hat{\lambda} = \vec{r}_{G/C'} \times m_{tot}\, \vec{a}_G \cdot \hat{\lambda} + \vec{\vec{I}} \cdot \dot{\vec{\omega}} \cdot \hat{\lambda} + \vec{\omega} \times \left(\vec{\vec{I}} \cdot \vec{\omega}\right) \cdot \hat{\lambda} \tag{10}$$

$\left(\vec{r}_{D/C} \times \vec{F}_D\right) \cdot \hat{\lambda}$ can be ignore. We can get rid of $\vec{F}_D$. For the total rotation,

$$\vec{\omega} = \dot{\psi}\, \hat{k} - \dot{\phi}\, \hat{\lambda}$$

$$\dot{\hat{\lambda}} = \dot{\psi}\, \hat{n}$$

$$\dot{\vec{\omega}} = \ddot{\psi}\, \hat{k} - \ddot{\phi}\, \hat{\lambda} - \dot{\phi}\, \dot{\hat{\lambda}} = \ddot{\psi}\, \hat{k} - \ddot{\phi}\, \hat{\lambda} - \dot{\phi}\, \dot{\psi}\, \hat{n}$$

For $\vec{r}_{G/C'}$ and $\vec{r}_{G/C}$,

$$\vec{r}_{G/C'} = \vec{r}_{G/C} = b\, \hat{\lambda} + h\, \hat{e}_r$$

For $\dot{\vec{H}}_{/G}$

$$\dot{\vec{H}}_{/G} = \frac{d}{dt}\left(\vec{\vec{I}} \cdot \vec{\omega}\right) = \vec{\vec{I}} \cdot \dot{\vec{\omega}} + \vec{\omega} \times \left(\vec{\vec{I}} \cdot \vec{\omega}\right) = (a)\, \hat{\lambda} + (b)\, \hat{n} + (c)\, \hat{k} \tag{11}$$

In Equation (11),

$$\begin{aligned} a = \dot{\psi}\, \left(\dot{\psi}\, \left(I_2\, \cos\phi\, \sin\phi - I_3\, \cos\phi\, \sin\phi\right) + I_{13}\, \dot{\phi}\, \sin\phi\right) - I_1\, \ddot{\phi} + I_{13}\, \ddot{\psi}\, \cos\phi \\ - I_{13}\, \dot{\phi}\, \dot{\psi}\, \sin\phi \end{aligned}$$

$$\begin{aligned} b = \dot{\phi}\, \left(\dot{\psi}\, \left(I_3 \cos\phi^2 + I_2 \sin\phi^2\right) - I_{13}\, \dot{\phi}\, \cos\phi\right) - \dot{\psi}\, \left(I_1\, \dot{\phi} - I_{13}\, \dot{\psi}\, \cos\phi\right) - I_{13}\, \ddot{\phi}\, \sin\phi \\ - \ddot{\psi}\, \left(I_2\, \cos\phi\, \sin\phi - I_3\, \cos\phi\, \sin\phi\right) - \dot{\phi}\, \dot{\psi}\, \left(I_2 \cos\phi^2 + I_3 \sin\phi^2\right) \end{aligned}$$

$$\begin{aligned} c = \dot{\phi}\, \left(\dot{\psi}\, \left(I_2\, \cos\phi\, \sin\phi - I_3\, \cos\phi\, \sin\phi\right) + I_{13}\, \dot{\phi}\, \sin\phi\right) + \ddot{\psi}\, \left(I_3 \cos\phi^2 + I_2 \sin\phi^2\right) \\ + \dot{\phi}\, \dot{\psi}\, \left(I_2\, \cos\phi\, \sin\phi - I_3\, \cos\phi\, \sin\phi\right) - I_{13}\, \ddot{\phi}\, \cos\phi \end{aligned}$$

Replace $\dot{\psi}$ and $\ddot{\psi}$ by Equation (3) and Equation (4) into Equation (6), we can get expression of $\vec{a}_G$ without $\dot{\psi}$ and $\ddot{\psi}$ term. Substitute $\vec{a}_G$ by new Equation (6) and substitute $\dot{\vec{H}}_{/G}$ by Equation (11) into Equation (10), we can get the final governing equation.

$$\begin{aligned} I_1\, \ddot{\phi} + h^2\, m\, \ddot{\phi} - g\, h\, m\, \sin\phi + \frac{I_2\, v^2\, \sin\phi}{l^2\, \cos\phi} - \frac{I_3\, v^2\, \sin\phi}{l^2\, \cos\phi} - \frac{I_{13}\, \dot{\delta}\, v}{l \cos\delta^2} - \frac{I_{13}\, \dot{v}\, \sin\delta}{l\, \cos\delta} \\ + \frac{h^2\, m\, v^2\, \sin\phi}{l^2\, \cos\phi} - \frac{I_2\, v^2\, \sin\phi}{l^2 \cos\delta^2\, \cos\phi} + \frac{I_3\, v^2\, \sin\phi}{l^2 \cos\delta^2\, \cos\phi} + \frac{h\, m\, v^2\, \sin\delta}{l\, \cos\delta} + \frac{b\, \dot{\delta}\, h\, m\, v}{l \cos\delta^2} \\ - \frac{h^2\, m\, v^2\, \sin\phi}{l^2 \cos\delta^2\, \cos\phi} + \frac{b\, h\, m\, \dot{v}\, \sin\delta}{l\, \cos\delta} - \frac{I_{13}\, \dot{\phi}\, v\, \sin\delta\, \sin\phi}{l\, \cos\delta\, \cos\phi} + \frac{b\, h\, m\, \dot{\phi}\, v\, \sin\delta\, \sin\phi}{l\, \cos\delta\, \cos\phi} = 0 \end{aligned} \tag{12}$$

Linearize the governing equation by

$$\sin\phi = \phi \;\; \cos\phi = 1 \;\; \sin\delta = \delta \;\; \cos\delta = 1 \;\; \sin\psi = \psi \;\; \cos\psi = 1$$

And then times $\frac{l}{mh}$. We can get

$$\delta\, v^2 + b\, \dot{\delta}\, v + b\, \delta\, \dot{v} - g\, l\, \phi + h\, l\, \ddot{\phi} - \frac{I_{13}\, \dot{\delta}\, v}{h\, m} - \frac{I_{13}\, \delta\, \dot{v}}{h\, m} + \frac{I_1\, l\, \ddot{\phi}}{h\, m} + b\, \delta\, \phi\, \dot{\phi}\, v - \frac{I_{13}\, \delta\, \phi\, \dot{\phi}\, v}{h\, m} = 0$$

Since δ and ϕ are assumed to be very small, ignore terms with $\delta \cdot \phi$, δ^2 and ϕ^2:

$$\delta\, v^2 + b\, \dot{\delta}\, v + b\, \delta\, \dot{v} - g\, l\, \phi + h\, l\, \ddot{\phi} - \frac{I_{13}\, \dot{\delta}\, v}{h\, m} - \frac{I_{13}\, \delta\, \dot{v}}{h\, m} + \frac{I_1\, l\, \ddot{\phi}}{h\, m} = 0$$

By Solving the equation, we get

$$\ddot{\phi} = \frac{g\, l\, \phi}{h\, l + \frac{I_1\, l}{h\, m}} - \frac{b\, \dot{\delta}\, v}{h\, l + \frac{I_1\, l}{h\, m}} - \frac{b\, \delta\, \dot{v}}{h\, l + \frac{I_1\, l}{h\, m}} - \frac{\delta\, v^2}{h\, l + \frac{I_1\, l}{h\, m}} + \frac{I_{13}\, \dot{\delta}\, v}{l\, m\, h^2 + I_1\, l} + \frac{I_{13}\, \delta\, \dot{v}}{l\, m\, h^2 + I_1\, l} \quad (13)$$

If considering the tilt angle of the steer axis ζ, the result will become

$$\ddot{\phi} = \frac{g\, l\, \phi}{h\, l + \frac{I_1\, l}{h\, m}} - \frac{\delta\, v^2\, \cos\zeta}{h\, l + \frac{I_1\, l}{h\, m}} - \frac{b\, \dot{\delta}\, v\, \cos\zeta}{h\, l + \frac{I_1\, l}{h\, m}} - \frac{b\, \delta\, \dot{v}\, \cos\zeta}{h\, l + \frac{I_1\, l}{h\, m}} + \frac{I_{13}\, \dot{\delta}\, v\, \cos\zeta}{l\, m\, h^2 + I_1\, l} + \frac{I_{13}\, \delta\, \dot{v}\, \cos\zeta}{l\, m\, h^2 + I_1\, l} \quad (14)$$

5 Accuracy of the simplified model

In order to test the accuracy of this simplified model, we compare it with the benchmark derived by Meijaard et al.(2007). We did a numerical comparison with a full non-linear simulation. In the full non-linear simulation, the forward speed is set to be a various constant. So all $\dot{v}$ terms need to be neglect. The benchmark is as follows.

$$\mathbf{M}\, \ddot{\mathbf{q}} + \mathbf{C_1}\, \dot{\mathbf{q}}\, v + \mathbf{q}\left(\mathbf{K_2}\, v^2 + \mathbf{K_0}\, g\right) = \mathbf{f} \quad (15)$$

Where $\mathbf{q} = [\phi, \delta]^T$ and $\mathbf{f} = [T_\phi, T_\delta]^T$. Since T_δ is steering torque, which will cause a clockwise (looking down) action of the handlebar assembly[1]. It would be the torque that a rider or a motor applies to the handlebar in our model. We cannot use the second equation in the benchmark because T_δ is unknown. Since T_ϕ is the right lean torque, which will cause the bicycle to lean[1]. We can assume T_ϕ to be 0. We can use the first equation of the benchmark. For $\mathbf{M}$, $\mathbf{K_0}$, $\mathbf{K_2}$ and $\mathbf{C_1}$,

$$\mathbf{M} = \begin{bmatrix} 80.81722 & 2.31941332208709 \\ 2.31941332208709 & 0.29784188199686 \end{bmatrix}$$

$$\mathbf{K_0} = \begin{bmatrix} -80.95 & -2.59951685249872 \\ -2.59951685249872 & -0.80329488458618 \end{bmatrix}$$

$$\mathbf{K_2} = \begin{bmatrix} 0 & 76.59734589573222 \\ 0 & 2.65431523794604 \end{bmatrix}$$

$$\mathbf{C_1} = \begin{bmatrix} 0 & 33.86641391492494 \\ -0.85035641456978 & 1.68540397397560 \end{bmatrix}$$

Substitute $\mathbf{M}$, $\mathbf{K_0}$, $\mathbf{K_2}$ and $\mathbf{C_1}$ into Equation (15), we can get

$$2.3194\,\ddot{\delta} + 80.817\,\ddot{\phi} - 80.95\,g\,\phi + 33.866\,\dot{\delta}\,v - 1.0\,\delta\,\left(2.5995\,g - 76.597\,v^2\right) = 0.0$$

Since we assume we can control the change rate of steering $\dot{\delta}$ directly, $\ddot{\delta}$ is assume to be 0. Solve the equation about $\ddot{\phi}$, and we get

$$\ddot{\phi} = -0.9478\,\delta\,v^2 - 0.419\,\dot{\delta}\,v + 0.03217\,\delta\,g + 1.002\,g\,\phi \tag{16}$$

For the simplified model, we use all the parameters based on the benchmark.

$$\begin{aligned}
\dot{v} &= 0 \\
l &= 1.02 \\
b &= 0.3 \\
h &= 0.9 \\
m &= 94 \\
I_1 &= 9.2 \\
I_{13} &= 2.4
\end{aligned}$$

Solve the equation about $\ddot{\phi}$, we can get

$$\ddot{\phi} = -0.9719\,\delta\,v^2 - 0.264\,\dot{\delta}\,v + 0.9913\,g\,\phi \tag{17}$$

Compare Equation (16) and Equation (17), the error of $\delta\,v^2$ term is 2.5% and the error of $g\,\phi$ term is 1.0%. There is no $\delta\,g$ term in the linearized equation of simplified model, but the coefficient of $\delta\,g$ term is 0.03, which is relative small. The error of $\dot{\delta}\,v$ term is 36%, which is relative large. The reason of this happening is that in the simplified model, the moment of inertia of the handlebar is not considered. So the effect of $\dot{\delta}$ is not considered. The simplified model will be more accurate when the bicycle is moving in relative low speed and the handlebar is not rotating very fast. If we use following conditions,

$$\begin{aligned}
g &= 9.81 \\
\phi &= 0.2 \\
\delta &= 0.2 \\
v &= 2 \\
\dot{\delta} &= 0.2
\end{aligned}$$

The result of $\ddot{\phi}$ is $\ddot{\phi} = 1.0619$ by Equation (17) and $\ddot{\phi} = 1.1025$ by Equation (16). The error is 3.7%.

If considering the tilt angle of the steer axis ζ and substitute $\zeta = \pi/10$, the equation of $\ddot{\phi}$ become

$$\ddot{\phi} = -0.9243\,\delta\,v^2 - 0.2511\,\dot{\delta}\,v + 0.9913\,g\,\phi \tag{18}$$

Compare Equation (18), Equation (16) and Equation (17). We found that considering the tilt angle of the steer axis ζ only let the $\delta\,v^2$ term become more accurate a little bit, but $\delta\, g$ term and $\dot{\delta}\, v$ term are still not accurate. The result of $\ddot{\delta}$ at low speed and low steering speed is $\ddot{\delta} = 1.1051$ by Equation (18) and the error is 0.2%.

Based on previous analysis, the tilt angle of the steer axis ζ can be ignore since it doesn't improve a lot on the accuracy of the whole dynamic model and the governing equation. All the following control algorithms are derived without the tilt angle of the steer axis ζ.

6 Controller design

Since there are multiple variables in the governing Equation (13), we implemented state space form to design the controller.

6.1 State space form

For state space form,

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t)$$

We assume the system is fully observable, so that $\mathbf{y}(t) = \mathbf{x}(t)$. Base on the equation,

$$\mathbf{x}(t) = \begin{bmatrix} \phi \\ \dot{\phi} \\ \delta \\ v \end{bmatrix}$$

$$\mathbf{u}(t) = \begin{bmatrix} \dot{\delta} \\ \dot{v} \end{bmatrix}$$

For doing linearization about nominal state of a system, assume nominal state is $\mathbf{x}_0$ and $\mathbf{u}_0$,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \approx \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0, t) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0, \mathbf{u}_0, t} (\mathbf{x} - \mathbf{x}_0) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\mathbf{x}_0, \mathbf{u}_0, t} (\mathbf{u} - \mathbf{u}_0) \tag{19}$$

In Equation (19), the equilibrium state is

$$\mathbf{x}_0 = \begin{bmatrix} \phi_0 \\ \dot{\phi}_0 \\ \delta_0 \\ v_0 \end{bmatrix}$$

$$\mathbf{u}_0 = \begin{bmatrix} \dot{\delta}_0 \\ \dot{v}_0 \end{bmatrix}$$

$\Delta\mathbf{x}(t)$ and $\Delta\mathbf{u}(t)$ are defined as

$$\Delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}_0$$

$$\Delta\mathbf{u}(t) = \mathbf{u}(t) - \mathbf{u}_0$$

So that the equation can be represent by

$$\Delta\dot{\mathbf{x}} = \mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(t)\Delta\mathbf{u}(t) \tag{20}$$

In Equation (20), $\mathbf{F}(t)$ and $\mathbf{G}(t)$ can be derived as

$$\mathbf{F}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g\,l}{h\,l+\frac{I_1\,l}{h\,m}} & 0 & \frac{I_{13}\,\dot{v}_0}{l\,m\,h^2+I_1\,l} - \frac{b\,\dot{v}_0}{h\,l+\frac{I_1\,l}{h\,m}} - \frac{{v_0}^2}{h\,l+\frac{I_1\,l}{h\,m}} & \frac{I_{13}\,\dot{\delta}_0}{l\,m\,h^2+I_1\,l} - \frac{2\,\delta_0\,v_0}{h\,l+\frac{I_1\,l}{h\,m}} - \frac{b\,\dot{\delta}_0}{h\,l+\frac{I_1\,l}{h\,m}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

$$\mathbf{G}(t) = \begin{bmatrix} 0 & 0 \\ \frac{I_{13}\,v_0}{l\,m\,h^2+I_1\,l} - \frac{b\,v_0}{h\,l+\frac{I_1\,l}{h\,m}} & \frac{I_{13}\,\delta_0}{l\,m\,h^2+I_1\,l} - \frac{b\,\delta_0}{h\,l+\frac{I_1\,l}{h\,m}} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{22}$$

6.2 Linear–quadratic regulator

Using LQR (linear–quadratic regulator), we can find the most optimal control to operate a dynamic system. It has minimum cost when the cost is described as a quadratic function.

6.2.1 Controller design

For the parameters and the equilibrium state,

$$l = 1.02$$
$$b = 0.3$$
$$h = 0.9$$
$$m = 94$$
$$I_1 = 9.2$$
$$I_{13} = 2.4$$
$$g = 9.81$$
$$\phi_0 = 0$$
$$\dot{\phi}_0 = 0$$
$$\delta_0 = 0$$
$$v_0 = 2$$
$$\dot{\delta}_0 = 0$$
$$\dot{v}_0 = 0$$

$$\mathbf{F} = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ 9.7249 & 0 & -3.8876 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 0 & 0 \\ -0.52799 & 0 \\ 1.0 & 0 \\ 0 & 1.0 \end{bmatrix}$$

For the cost function, the quadratic cost function is defined as

$$J = \int_{t_0}^{t_1} \left(\Delta\mathbf{x}^T \mathbf{Q}\, \Delta\mathbf{x} + \Delta\mathbf{u}^T \mathbf{R}\, \Delta\mathbf{u} \right) dt$$

Let two coefficient matrix be identity matrix,

$$\mathbf{Q} = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix}$$

Solve the continuous-time algebraic Riccati equation,

$$\mathbf{F}^T \mathbf{x} + \mathbf{x}\mathbf{F} - \mathbf{x}\mathbf{G}\mathbf{R}^{-1}\mathbf{G}^T \mathbf{x} + \mathbf{Q} = 0$$

Solution is

$$\mathbf{C} = \begin{bmatrix} -14.77 & -4.8301 & 4.8274 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

Based on the physics analysis of the bicycle by Meijaard et al.(2007), there is no first-order change in speed due to lean, and speed change alone cannot cause lean[1]. We could design two decoupled controllers for speed and steering control. This gain matrix **C** accord with this demand. The input for the controller is

$$\Delta\mathbf{u} = -\mathbf{C}\Delta\mathbf{x} \tag{23}$$

6.2.2 Evaluation: recovery from leaning

To evaluate the linear feedback controller, we can first test whether it can recover a bicycle from a leaning state to the upright equilibrium state by traversal of different initial condition with different lean angle and steering angle. The criterion for recovery failure is that steering angle or lean angle is greater than 1 rad, or after 10 seconds the absolute value of lean angle is still greater than 0.01. See Figure 5 for result.

As is shown in Figure 5, the LQR controller cannot recover the bicycle from leaning state to upright position when the rear wheel speed is lower than 1.8 m/s. The LQR controller starts being able to recover the bicycle from leaning state to upright position after the rear wheel speed exceed 1.8 m/s, and the maximum lean angle reach 0.3 rad when the speed of rear wheel is 1.91 m/s. After the rear wheel speed exceed 2 m/s, the maximum lean angle growth gradually and the growing speed of maximum lean angle is decreasing. The simplified model will be more accurate at lower speed and smaller angle.

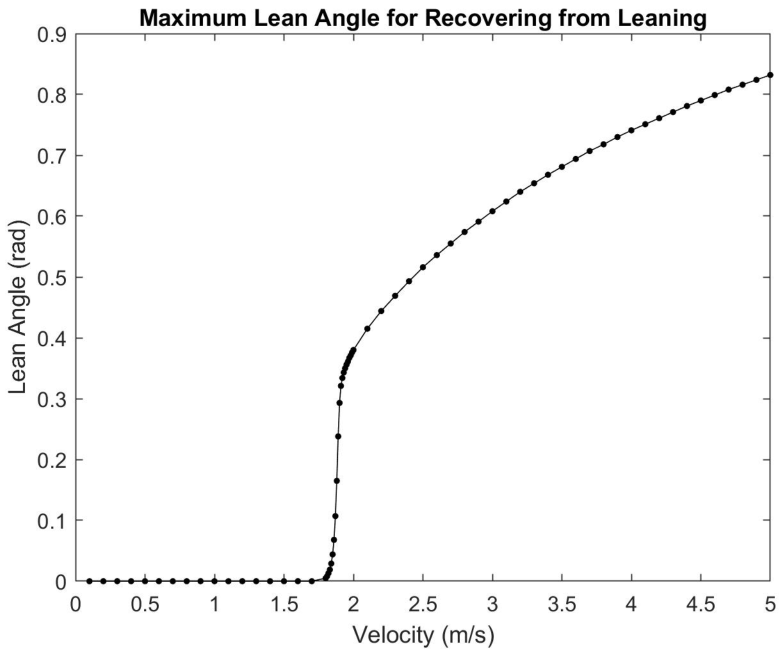

Figure 5: Maximum Lean Angle for Recovering from Leaning

6.2.3 Evaluation: steady turn

To keep the bicycle doing a steady turn, since we derived the state space form from linearized governing Equation (13), we cannot calculate a new gain matrix based on the new equilibrium state. So we still use the same gain matrix to test the maximum lean angle and steering angle for the bicycle to do a steady turn. The criteria for failing to keep the bicycle doing a steady turn is that the difference between the actual lean angle and the desired lean angle is greater than 0.1, or after 10 seconds the change rate of lean angle is still greater than 0.000001. See Figure 6 for result.

As is shown in Figure 6, the LQR controller cannot keep the bicycle doing a steady turn with rear wheel speed less than 1.8 m/s. The LQR controller starts being able to keep the bicycle doing a steady turn with rear wheel speed greater than 1.8 m/s. The maximum lean angle of steady turn keep increasing when the rear wheel speed increasing. For the maximum steering angle to keep the bicycle to do a steady turn, it reach its maximum at 2.4 m/s. Because a large steering angle will cause a even larger increase of lean angle with a factor of v^2, the maximum steering angle start decreasing after speed exceed 2.4 m/s.

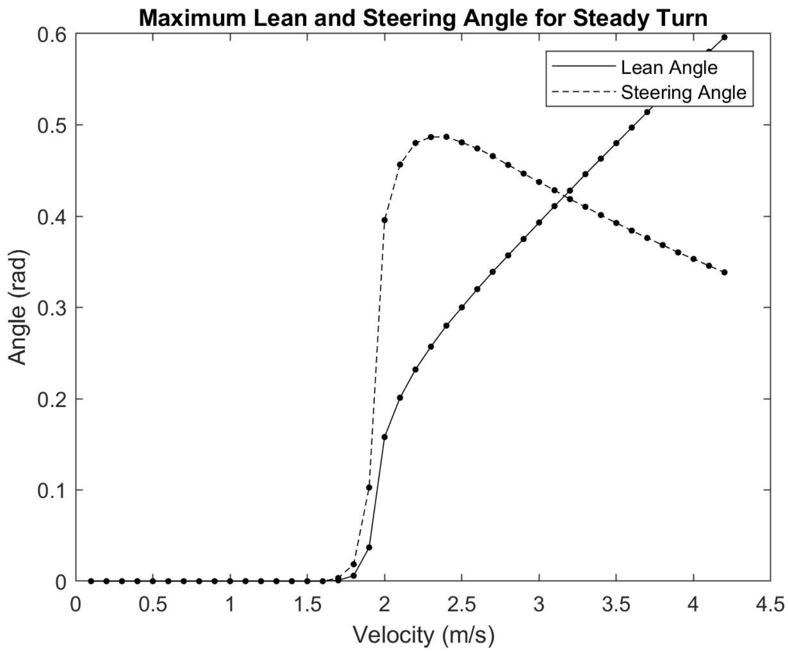

Figure 6: Maximum Lean and Steering Angle for Steady Turn

6.2.4 Evaluation: robustness

When the observation of the state of the bicycle is inaccurate, the controller may fail due to the offset. The robustness of the controller become very important. The criteria for failing to recover is that the steering angle or the lean angle is greater than 1 rad, or after 10 seconds the absolute value of lean angle change rate is still greater than 0.01. The rear wheel speed is 2 m/s. See Figure 7 for result.

As is shown in Figure 7, the robustness of the LQR controller is not very good. The controller cannot balance the bicycle if the error of lean angle exceed 0.04 rad. When the lean angle is accurate, the maximum error for steering angle is 0.13 rad.

6.3 Sliding mode control

In order to solve the problems in LQR controller, a non-linear controller, sliding mode control, was implemented. In control systems, sliding mode control, or SMC, is a nonlinear control method that alters the dynamics of a nonlinear system by application of a discontinuous control signal that forces the system to "slide" along a cross-section of the system's normal behavior[4]. For Lyapunov

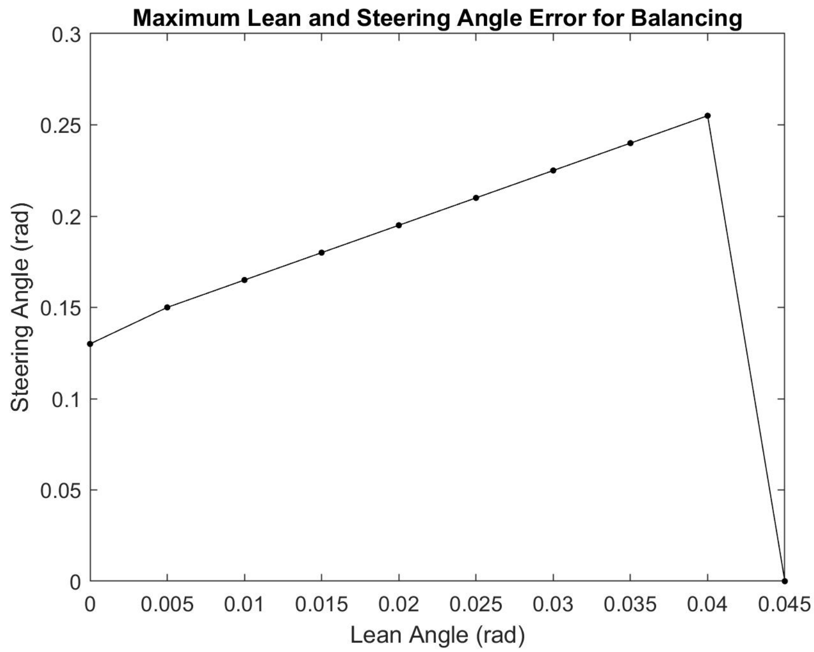

Figure 7: Maximum Lean and Steering Angle Error for Balancing

Stability[5]:

$$V(\mathbf{x}) = 0 \text{ if and only if } x = 0$$

$$V(\mathbf{x}) > 0 \text{ if and only if } x \neq 0$$

$$\dot{V}(\mathbf{x}) < 0 \text{ for all values of } x \neq 0$$

Consider a plant with single input u:

$$s(\mathbf{x}) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

If we choose feedback control law u so that

$$\dot{s}(\mathbf{x}) < 0 \text{ when } s(\mathbf{x}) > 0$$

and

$$\dot{s}(\mathbf{x}) > 0 \text{ when } s(\mathbf{x}) < 0$$

We can find $V(x)$ as:

$$V(\mathbf{x}) = \frac{1}{2} s^2(\mathbf{x})$$

$$\dot{V}(\mathbf{x}) = s^T(\mathbf{x}) \cdot \dot{s}(\mathbf{x}) < 0$$

The system is stable. s $(\mathbf{x})$ is called sliding surface. To design the controller u, we can first solve the equation $\dot{s}(\mathbf{x}) = 0$ and solve the u as u_{eq}. After that we can add a u^{vss} term to u_{eq} as the finial input u to ensure $\dot{V}(x) = s^T(\mathbf{x}) \cdot \dot{s}(\mathbf{x}) < 0$. u^{vss} stands for Variable Structure System. Ideally,

$$u^{vss} = u_0 \cdot sgn(s(\mathbf{x}))$$

In order to be easily implemented, use the following expression to replace sign function:

$$u^{vss} = k^{vss}\xi(s(\mathbf{x}))$$

Where

$$\xi(s) = \frac{s(\mathbf{x})}{|s(\mathbf{x})| + \Delta}$$

Δ is a very small positive constant.

6.3.1 Controller design

Since there is no first-order change in speed due to lean, also speed change alone cannot cause lean as well[1], We designed a decoupled controllers for steering using sliding mode control. Speed control can be design by another simple controller. For the error between the current state and desired state is e:

$$e_1 = \hat{x}_1 - x_1$$

$$e_2 = \hat{x}_2 - x_2$$

$$s(\mathbf{x}) = c_1 e_1 + c_2 e_2$$

$$x_2 = \dot{x_1}$$

$$s(\mathbf{x}) = c_1 e_1 + c_2 \dot{e_1}$$

Let $\frac{c_1}{c_2} = c$,

$$s(\mathbf{x}) = c e_1 + \dot{e_1}$$

$$\dot{s}(\mathbf{x}) = c\dot{e_1} + \ddot{e_1}$$

$$\hat{u} = u^{eq} + u^{vss}$$

$$u^{vss} = k^{vss}\xi(s)$$

$$\xi(s) = \frac{s}{|s| + \delta}$$

So for the lean angle ϕ and desired lean angle $\hat{\phi}$,

$$e_\phi = \hat{\phi} - \phi$$

$$s_\phi = \dot{e_\phi} + c_\phi e_\phi$$

$$\dot{s}_\phi = \ddot{\hat{\phi}} - \ddot{\phi} + c_\phi \dot{e_\phi} = 0 \tag{24}$$

By solving Equation (12), we can get $\ddot{\phi}$ as

$$
\begin{aligned}
\ddot{\phi} = & \frac{I_{13}\,\dot{v}\,\sin\delta}{l\,m\,\cos\delta\,h^2 + I_1\,l\,\cos\delta} - \frac{I_2\,v^2\,\sin\phi}{m\,\cos\phi\,h^2\,l^2 + I_1\,\cos\phi\,l^2} + \frac{I_3\,v^2\,\sin\phi}{m\,\cos\phi\,h^2\,l^2 + I_1\,\cos\phi\,l^2} \\
& + \frac{I_{13}\,\dot{\delta}\,v}{l\,m\,h^2\cos\delta^2 + I_1\,l\cos\delta^2} + \frac{I_2\,v^2\,\sin\phi}{m\,\cos\phi\,h^2\,l^2\cos\delta^2 + I_1\,\cos\phi\,l^2\cos\delta^2} \\
& - \frac{I_3\,v^2\,\sin\phi}{m\,\cos\phi\,h^2\,l^2\cos\delta^2 + I_1\,\cos\phi\,l^2\cos\delta^2} + \frac{g\,h\,m\,\sin\phi}{m\,h^2 + I_1} - \frac{h\,m\,v^2\,\sin\delta}{l\,m\,\cos\delta\,h^2 + I_1\,l\,\cos\delta} \\
& - \frac{h^2\,m\,v^2\,\sin\phi}{m\,\cos\phi\,h^2\,l^2 + I_1\,\cos\phi\,l^2} + \frac{h^2\,m\,v^2\,\sin\phi}{m\,\cos\phi\,h^2\,l^2\cos\delta^2 + I_1\,\cos\phi\,l^2\cos\delta^2} \\
& - \frac{b\,h\,m\,\dot{v}\,\sin\delta}{l\,m\,\cos\delta\,h^2 + I_1\,l\,\cos\delta} + \frac{I_{13}\,\dot{\phi}\,v\,\sin\delta\,\sin\phi}{l\,m\,\cos\delta\,\cos\phi\,h^2 + I_1\,l\,\cos\delta\,\cos\phi} \\
& - \frac{b\,\dot{\delta}\,h\,m\,v}{l\,m\,h^2\cos\delta^2 + I_1\,l\cos\delta^2} - \frac{b\,h\,m\,\dot{\phi}\,v\,\sin\delta\,\sin\phi}{l\,m\,\cos\delta\,\cos\phi\,h^2 + I_1\,l\,\cos\delta\,\cos\phi}
\end{aligned}
\tag{25}
$$

Substitute Equation (25) into Equation (24), we can get the finial controller u_ϕ as

$$u_\phi^{eq} = \dot{\delta}$$

$$\hat{u} = u_\phi^{eq} + k_\phi^{vss}\xi\left(s_\phi\right)$$

$$\xi\left(s_\phi\right) = \frac{s}{|s| + \Delta_\phi}$$

Where k_ϕ and Δ_ϕ are two constant. In our implementation, $k_\phi = -30$, $\Delta_\phi = 1$ and $c_\phi = 100$.

6.3.2 Evaluation: recovery from leaning

Similar to LQR, we can evaluate the controller by whether it can recover form leaning position. The criteria for failing to recover is that steering angle or lean angle is greater than 1 rad, or after 10 seconds the absolute value of lean angle is still greater than 0.01. See Figure 8 for result.

As is shown in Figure 8, the controller starts to be able to recover the bicycle from leaning position when the speed is greater than 0.2 m/s. The maximum lean angle for recovering from leaning increases as the rear wheel speed increasing. The performance is better than LQR.

6.3.3 Evaluation: steady turn

Similar to LQR, we can evaluate the controller by whether it can keep the bicycle to do a steady turn. The criteria for failing to keep steady turn is that the difference between the actual lean angle and the desired lean angle is greater than 0.1, or the steering angle or the lean angle is greater than 1 rad, or after 10 seconds the change rate of lean angle rate is still greater than 0.000001. See Figure 9 for result.

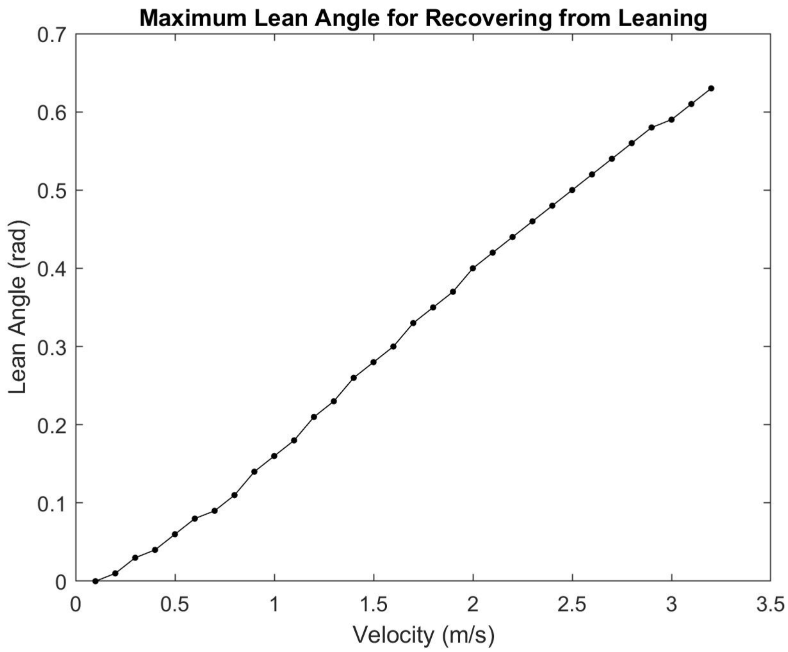

Figure 8: Maximum Lean Angle for Recovering from Leaning

As is shown in Figure 9, the controller starts to keep the bicycle to do a steady turn after the rear wheel speed exceed 0.3 m/s. The maximum lean angle for steady turn is keep increasing as the rear wheel speed rising. The maximum steering angle overshoots the upper boundary, 1 rad, and it starts decreasing as the rear wheel speed rising. The performance is better than LQR.

6.3.4 Evaluation: robustness

Similar to LQR, we can evaluate the controller by whether it can still keep balanced when the state observation is not accurate. The criteria for failing to recover is that the steering angle or the lean angle is greater than 1 rad, or after 10 seconds the absolute value of lean angle change rate is still greater than 0.01. The rear wheel speed is 2 m/s. See Figure 10 for result.

As is shown in Figure 10, the bicycle can still be balanced when the error of lean angle error is 0.33 rad. The error of steering angle has almost no effect when the error is less than 0.4. The bicycle can almost not be balanced when the error is greater than 0.6 rad. The robustness is much better than LQR controller.

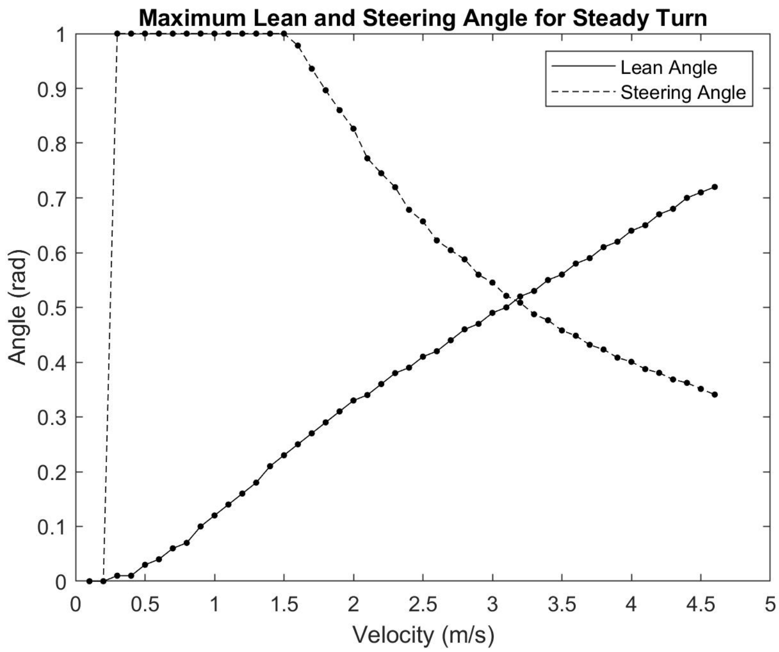

Figure 9: Maximum Lean and Steering Angle for Steady Turn

6.4 Simulation and animation

We simulated and animated the bicycle in MATLAB. See Figure 11 for animation result. All the MATLAB codes can be found at github.com/zhidiyang.

6.5 Conclusion for controllers

Based on the evaluation of two controllers, the non-linear controller, sliding mode control, is superior to the LQR controller. It can control the bicycle to recover to upright position or keep the bicycle doing a steady turn, at a relative lower speed. The robustness of it is much better than the LQR controller, which is very important in real life implementation. But the sliding mode control controller needs higher commanding rate for the change rate of steering and it requires better motor. Also the non-linear controller is designed based on this simplified model and simulated in the same model. The result will not as good as current result if we test it in the benchmark. The influence of delay in the system is not considered and it might also influence the robustness of the controllers.

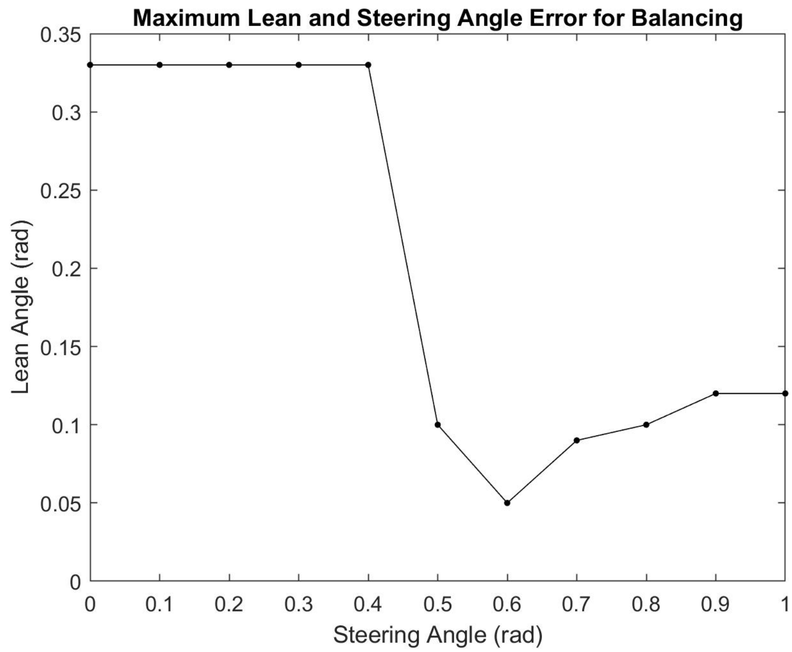

Figure 10: Maximum Lean and Steering Angle Error for Balancing

7 Conclusion

This article derives a simplified bicycle model, which can be used to design a linear or non-linear controller for autonomous bicycle. We compared the accuracy of the model with Meijaard et al.(2007)'s benchmark. The result shows that the simplified model accuracy is acceptable when the bicycle is moving in relative low speed and the handlebar is not steering very fast. We implemented two controllers in simulations and tested the performance and robustness of it. The sliding mode control controller is superior than LQR on performance and robustness, but it needs more agile commands and it can be more computationally expensive.

A Front wheel direction with ζ

Assume the radius of the wheel is r. l is the distance between front and rear wheel. Find two point P and Q. P is the center of front wheel. Q is another point outside the plane of the front wheel. $PQ\perp$the plane of the front wheel. r is the trail of the front wheel. I define it as the distance between P and the point on the front fork, which has same height as point P. Point C is the rotation center of point P and Q. For point P and Q, $x_P\, y_P\, z_P$ and $x_Q\, y_Q\, z_Q$ is the

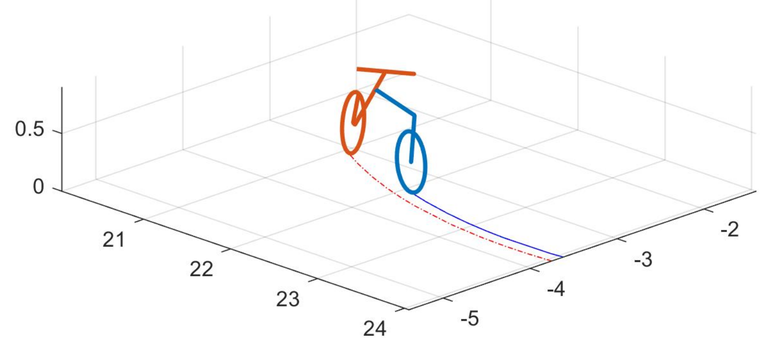

Figure 11: Simulation and Animation in MATLAB

coordinate of two points. Use **X** to represent the coordinate of two points. Use **C** to represent the coordinate the rotation center.

$$\mathbf{X} = \begin{bmatrix} x_P & x_Q \\ y_P & y_Q \\ z_P & z_Q \end{bmatrix}$$

So based on the spatial relationship,

$$\mathbf{X} = \begin{bmatrix} l & l \\ 0 & 1 \\ r & r \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} l - c & l - c \\ 0 & 0 \\ r & r \end{bmatrix}$$

For steering, the rotation axis,

$$\hat{n} = \begin{bmatrix} -\sin\zeta \\ 0 \\ \cos\zeta \end{bmatrix}$$

The rotation matrix $\underline{\underline{\mathbf{R}}}$,

$$\underline{\underline{\mathbf{R}}} = \begin{bmatrix} \sin\zeta^2 - \cos\delta\left(\sin\zeta^2 - 1\right) & -\sin\delta\,\cos\zeta & \cos\delta\,\cos\zeta\,\sin\zeta - \cos\zeta\,\sin\zeta \\ \sin\delta\,\cos\zeta & \cos\delta & \sin\delta\,\sin\zeta \\ \cos\delta\,\cos\zeta\,\sin\zeta - \cos\zeta\,\sin\zeta & -\sin\delta\,\sin\zeta & \cos\zeta^2 - \cos\delta\left(\cos\zeta^2 - 1\right) \end{bmatrix}$$

$$\mathbf{X}' = \underline{\underline{\mathbf{R}}} \cdot (\mathbf{X} - \mathbf{C}) + \mathbf{C} =$$

$$\begin{bmatrix} l - c\cos\zeta^2 + c\,\cos\delta\cos\zeta^2 & l - \sin\delta\,\cos\zeta - c\cos\zeta^2 + c\,\cos\delta\cos\zeta^2 \\ c\,\sin\delta\,\cos\zeta & \cos\delta + c\,\sin\delta\,\cos\zeta \\ r - c\,(\cos\zeta\,\sin\zeta - \cos\delta\,\cos\zeta\,\sin\zeta) & r - \sin\delta\,\sin\zeta - c\,(\cos\zeta\,\sin\zeta - \cos\delta\,\cos\zeta\,\sin\zeta) \end{bmatrix}$$

For leaning, the rotation matrix $\underline{\underline{\mathbf{R}}}$,

$$\underline{\underline{\mathbf{R}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$\mathbf{X}'' = \underline{\underline{\mathbf{R}}} \cdot \mathbf{X}'$$

$$\overrightarrow{P''Q''} = \begin{bmatrix} \sin\delta\,\cos\zeta \\ \sin\delta\,\sin\phi\,\sin\zeta - \cos\delta\,\cos\phi \\ \cos\delta\,\sin\phi + \cos\phi\,\sin\delta\,\sin\zeta \end{bmatrix}$$

Since $\overrightarrow{P''Q''}$ is perpendicular to the front wheel,

$$\overrightarrow{P''Q''} \perp DF$$

$$\overrightarrow{P''Q''} \cdot \overrightarrow{DF} = 0$$

$$\tan\alpha = \frac{\sin\delta\,\cos\zeta}{\cos\delta\,\cos\phi - \sin\delta\,\sin\phi\,\sin\zeta}$$

If $\zeta = 0$,

$$\tan\alpha = \frac{\sin\delta}{\cos\delta\,\cos\phi}$$

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